An Approach to Evaluating the Prior Distribution of Weibull Parameters

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Abstract

The paper considers a new procedure for the Bayesian estimation of the Weibull distribution. The suggested procedure is based on prior information available at two points of the exposure variable as the estimates of the cumulative distribution function (CDF) and its standard deviation. As a result, the procedure provides joint prior and posterior distributions of the Weibull parameters and the posterior estimates of the mean and standard deviation of the CDF at any given value of the exposure variable. A numeric example is discussed as an illustration.

1. Introduction

Weibull is one of the most widely used distributions in reliability and risk assessment (RRA). Practical use of the Bayes' estimation is often associated with difficulties related to elicitation of prior information and its formalization into the respective prior distribution. Opposite to the binomial and exponential distributions, which are also popular in RRA, the two-parameter Weibull distribution requires a two-dimensional joint prior distribution. It should be noted that in the framework of the Bayesian approach, both the exponential and binomial distributions, have their respective conjugate prior distributions, which makes their practical use more convenient.

The situation with the Weibull distribution is different. In the most realistic case, when both parameters of the distribution are considered as random variables, the fundamental result obtained by Soland (1966) states that the Weibull distribution does not have a conjugate *continuous* joint prior distribution.

Later, Soland (1969) proved that a conjugate *continuous-discrete* joint prior distribution exists for the Weibull distribution parameters. The continuous "component" of this distribution is related to the scale parameter of Weibull distribution (similar to Bayesian estimation of the exponential distribution, for which gamma is used as the conjugate prior distribution), and the discrete one – to the shape parameter. Although of a great academic interest, this result may be challenging to apply to real life problems due to the difficulties of evaluating prior information needed (mostly related to the shape parameter). A reliability application example of this approach can be found in Martz and Waller (1982).

In the sequel, we consider a new procedure for the Bayesian estimation of the Weibull distribution, which allows constructing *continuous* joint prior distribution of Weibull parameters as well as the posterior estimates of the mean and standard deviation of the estimated CDF (or the reliability function) at any given value of the exposure variable.

2. Procedure

Consider the prior information, which is available at two values (t_1 and t_2, $t_2 > t_1$) of the exposure variable as the estimates of CDF, $\hat{F}(t_k)$, and its standard deviation, $\hat{\sigma}\{\hat{F}(t_k)\}$, see Figure 1. This prior information can be divided in the following two cases. In the first case, it comes in the traditional form as life data sufficient for the Kaplan-Meier and Greenwood estimators to be used. It might be attributed to the empirical Bayesian inference. In the second case, the prior information is obtained using the expert elicitation methods (e.g., Ayyub, 2002).

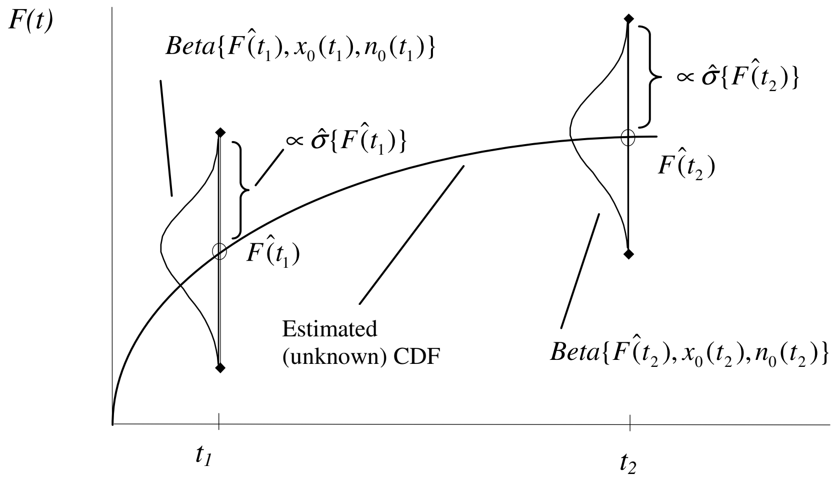

Figure 1: Prior information and Beta distributions
approximating uncertainty of the estimated CDF at a fixed exposure, t_k.

Based on these prior data, we are interested in obtaining the following:

- joint *prior* distribution of the Weibull distribution parameters;
- prior point estimates of Weibull parameters as the location of the *highest posterior probability density* (mode) of their joint distribution;
- *prior* distribution of the estimated CDF at any fixed exposure t_k;
- joint *posterior* distribution of the Weibull distribution parameters (based on data observed at t_3);
- posterior point estimates of the Weibull parameters as the location of the *highest posterior probability density* (mode) of their joint distribution, similar to the highest posterior density (HPD) estimates considered in Box and Tiao (1973);
- *posterior* distribution of the estimated CDF at exposure t_3 (based on data observed at t_3);

2.1 Prior Distribution

The Beta prior distribution with the following probability density function is assumed for the respective estimates of the CDF at a fixed exposure:

$$f(p) = \frac{\Gamma(n_0)}{\Gamma(x_0)\Gamma(n_0 - x_0)} p^{x_0 - 1}(1-p)^{n_0 - x_0 - 1}, \quad n_0 > 0, n_0 - x_0 > 0 \qquad (1)$$

where: p is the random variable representing the estimated CDF at a fixed exposure, $F(t_k)$; n_0, x_0 are the parameters of the Beta distribution, both positive quantities; $\Gamma(.)$ is the gamma function.

Based on the *prior* estimates of the CDF at a fixed exposure, $\hat{F}(t)$, $t=const,$ and its standard deviation, $\hat{\sigma}\{\hat{F}(t)\}$, the parameters of the respective *prior* beta-distribution can be obtained through the method of moments (Modaress, et. al, 1999) as follows:

$$x_0(t) = \frac{\hat{F}(t)[1-\hat{F}(t)]}{\hat{\sigma}^2\{\hat{F}(t)\}}, \quad n_0(t) = \frac{x_0(t)}{\hat{F}(t)} \tag{2}$$

By random sampling from the estimated Beta distributions, one can obtain pairs of random realizations of the estimated CDF at exposures t_1 and t_2: $\{F_i(t_1), F_i(t_2)\}$, $i = 1,2, ..., n$. For any such pair, as long as $F(t_1) < F(t_2)$, the *shape* and the *scale* parameters of the respective Weibull CDF can be obtained as:

$$\begin{cases} \beta = \dfrac{F^*(t_2) - F^*(t_1)}{Ln(t_2/t_1)} \\ \alpha = Exp\left(Ln(t_1) - \dfrac{F^*(t_1)}{\beta} \right) \end{cases} \tag{3}$$

$$here: F^*(t_k) = Ln(Ln(1-F(t_k))^{-1}), k = 1,2.$$

It is obvious that n simulated pairs of $\{F_i(t_1), F_i(t_2)\}$ would define n pairs of Weibull distribution parameters $\{\alpha_i, \beta_i\}$, which are used to construct their joint prior distribution.

Further, the obtained realizations of the Weibull CDF, $F_i(t; \alpha_i, \beta_i)$, can be extrapolated to a given exposure, say, t_3, to obtain the distribution of the CDF at t_3. Using (2), this distribution can again be approximated by the Beta-*prior* distribution with parameters $x_0(t_3)$ and $n_0(t_3)$.

2.2 Posterior Distributions

2.2.1 Posterior distribution of Weibull CDF a fixed exposure

The estimated Beta-prior at t_3 is now available for Bayesian estimation. Assume that observation data are available for exposure t_3 in the binary form of r failures out of N trials. Then, the *posterior* distribution is also the Beta distribution with parameters:

$$x_0(t_3)^* = x_0(t_3)+r, \; n_0(t_3)^* = n_0(t_3)+N \tag{4}$$

2.2.2 Posterior distribution of Weibull distribution parameters

The joint posterior distribution of the Weibull parameters can be numerically obtained by sampling random realizations from the estimated Beta distributions at exposures t_1, t_2 and (the newly obtained) t_3 and then using the probability paper method to obtain the sample of estimates of the Weibull distribution parameters.

3. Numeric example

Consider expert estimates of the failure probability of a mechanical component at t_1=1 and t_2=3 years of operation as $\hat{F}(t_1) = 0.01$ and $\hat{F}(t_2) = 0.05$, with 10% error. Treating the error as the coefficient of variation, the estimates of the standard deviation of the failure probability at the two exposures are $\hat{\sigma}\{\hat{F}(t_1)\} = 0.001$ and $\hat{\sigma}\{\hat{F}(t_2)\} = 0.005$, respectively. Using (2), the estimates of the Beta distribution parameters at the two exposures are $\{x_0(t_1) = 99, n_0(t_1) = 9899\}$ and $\{x_0(t_2) = 95, n_0(t_2) = 1899\}$, respectively. By sampling random realizations (n = 10000) of the CDF from the two estimated Beta distributions and the subsequent use of (3), one obtains the joint prior distribution of Weibull parameters shown in Figure 2. The prior estimates of the Weibull parameters corresponding to the mode of their joint distribution are found as $\{\beta_{pr} = 1.51, \alpha_{pr} = 22.95\}$. The failure probability and its standard deviation at a future exposure, t_3=5 years of operation, are estimated as 0.105 and 0.016, respectively, which using (2), can be translated into respective Beta distribution parameters at t_3: $\{x_0(t_3) = 40, n_0(t_3) = 386\}$.

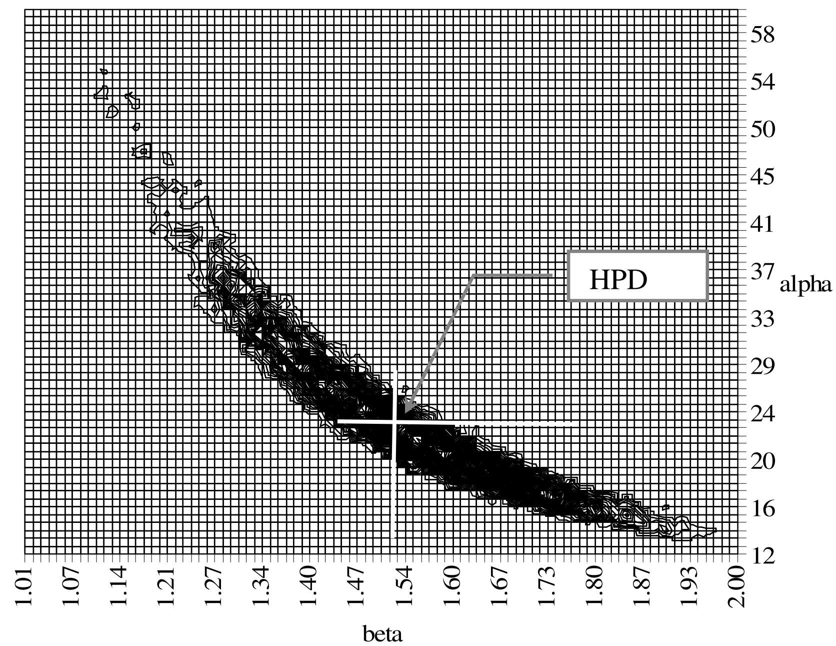

Figure 2: Contours of Joint Prior Distribution of the Weibull Parameters.

Further consider the data observed at t_3, which are 1 failure out of 12 components observed. The parameters of the posterior Beta distribution parameters at t_3, according to (4) are found as $\{x_0(t_3)^* = 41, n_0(t_3)^* = 398\}$. Figure 3 shows the joint posterior distribution of the Weibull distribution parameters obtained by sampling random realizations (n = 10000) from the estimated Beta distributions at exposures t_2 and t_3 and the subsequent use of (3). The posterior HPD estimates of Weibull parameters are found as $\{\beta_{post} = 1.70, \alpha_{post} = 16.55\}$. The posterior estimates of the failure probability (CDF) and its standard deviation at t_3 are obtained as 0.104 and 0.069. The estimation results are summarized in Table 1.

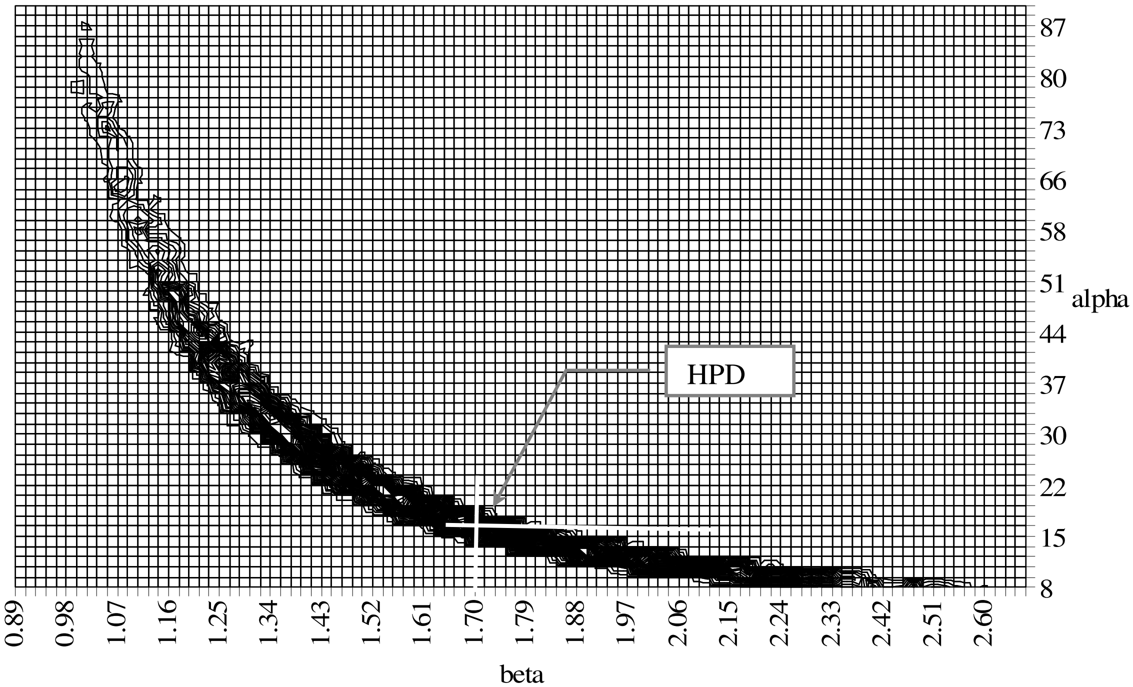

Figure 3: Contours of Joint Posterior Distribution of the Weibull Parameters.

Table 1: Prior and Posterior Estimates.

	Prior			Posterior
	$t_1 = 1$ (given)	$t_2=3$ (given)	$t_3=5$ (inferred)	$t_3=5$ (inferred)
$F(t)$	0.010	0.050	0.105	0.104
$\sigma\{F(t)\}$	0.001	0.005	0.016	0.069
α	22.95			16.55
β	1.51			1.70

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